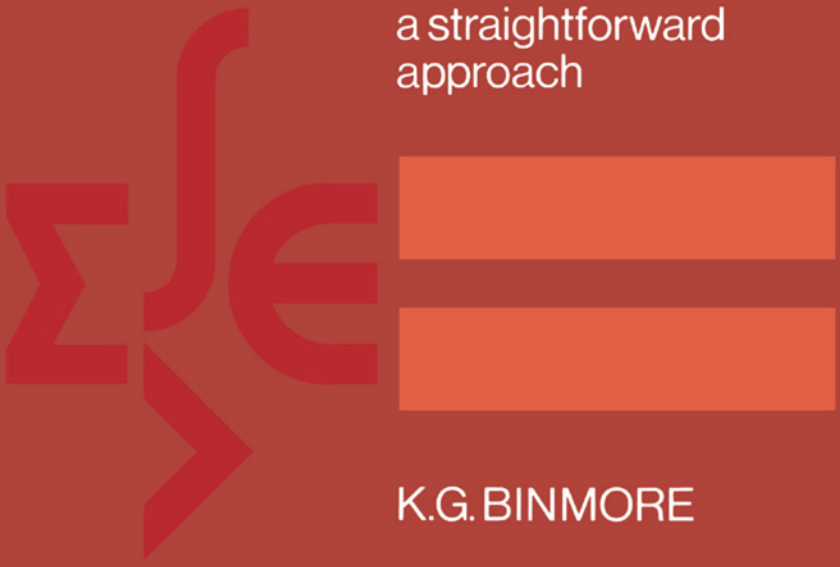
Mathematical Analysis

a straightforward approach



K.G. BINMORE

SECOND EDITION

CAMBRIDGE

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MATHEMATICAL ANALYSIS

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A STRAIGHTFORWARD APPROACH

K. G. BINMORE

Professor of Mathematics
London School of Economics and Political Science

SECOND EDITION

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PREFACE TO THE FIRST EDITION

This book is intended as an easy and unfussy introduction to mathematical analysis. Little formal reliance is made on the reader's previous mathematical background, but those with no training at all in the elementary techniques of calculus would do better to turn to some other book.

An effort has been made to lay bare the bones of the theory by eliminating as much unnecessary detail as is feasible. To achieve this end and to ensure that all results can be readily illustrated with concrete examples, the book deals only with 'bread and butter' analysis on the real line, the temptation to discuss generalisations in more abstract spaces having been reluctantly suppressed. However, the need to prepare the way for these generalisations has been kept well in mind.

It is vital to adopt a systematic approach when studying mathematical analysis. In particular, one should always be aware at any stage of what may be assumed and what has to be proved. Otherwise confusion is inevitable. For this reason, the early chapters go rather slowly and contain a considerable amount of material with which many readers may already be familiar. To neglect these chapters would, however, be unwise.

The exercises should be regarded as an integral part of the book. There is a great deal more to be learned from attempting the exercises than can be obtained from a passive reading of the text. This is particularly the case when, as may frequently happen, the attempt to solve a problem is unsuccessful and it is necessary to turn to the solutions provided at the end of the book.

To help those with insufficient time at their disposal to attempt all the exercises, the less vital exercises have been marked with the symbol †. (The same notation has been used to mark one or two passages in the text which can be omitted without great loss at a first reading.) The symbol * has been used to mark exercises which are more demanding than most but which are well worth attempting.

The final few chapters contain very little theory compared with the number of exercises set. These exercises are intended to illustrate the power of the techniques introduced earlier in the book and to provide the opportunity of some revision of these ideas.

This book arises from a course of lectures in analysis which is given at the London School of Economics. The students who attend this course are mostly not specialist mathematicians and there is little uniformity in their previous

mathematical training. They are, however, quite well-motivated. The course is a 'one unit' course of approximately forty lectures supplemented by twenty informal problem classes. I have found it possible to cover the material of this book in some thirty lectures. Time is then left for some discussion of point set topology in simple spaces. The content of the book provides an ample source of examples for this purpose while the more general theorems serve as reinforcement for the theorems of the text.

Other teachers may prefer to go through the material of the book at a more leisurely pace or else to move on to a different topic. An obvious candidate for further discussion is the algebraic foundation of the real number system and the proof of the Continuum Property. Other alternatives are partial differentiation, the complex number system or even Lebesgue measure on the line.

I would like to express my gratitude to Elizabeth Boardman and Richard Holmes for reading the text for me so carefully. My thanks are also due to 'Buffy' Fennelly for her patience and accuracy in preparing the typescript. Finally, I would like to mention M.C. Austin and H. Kestelman from whom I learned so much of what I know.

July 1976 K.G.B.

PREFACE TO THE SECOND EDITION

It is a pleasure to write a preface for the second edition of *Mathematical Analysis: A Straightforward Approach.* The first edition was well-received and I have therefore thought it wise to leave its text substantially unaltered except for one or two minor points of clarification and the correction of misprints. The major change is the addition of two long chapters on analysis in vector spaces for which there has been a considerable demand. These get as far as the idea of a derivative as a matrix and the use of the second order derivative of a real-valued function in classifying stationary points. More advanced material than this would seem to me better delayed until after the basic topological notions have been mastered. As far as the material covered is concerned, it does not involve the proof of many theorems and the necessary proofs involve no new analytic ideas. However, the material does require a certain facility with algebraic and geometric ideas and students with only a very limited knowledge of linear algebra may find it heavy going in spite of the fact that some discussion of the necessary concepts from linear algebra is included where appropriate. Another innovation is the inclusion of a collection of further problems for which the solutions are not given. I am grateful to John Erdös for some of these as well as other helpful suggestions. Teachers using this book as part of a taught course may find these problems helpful in setting work but I hope that they will not distract attention from the importance of working carefully through the exercises given in the main body of the text.

Finally, I would like to express my appreciation to those who have commented favourably on the first edition and to Mimi Bell for her patient help in preparing the typescript for the second edition.

October 1981 K.G.B.

1 REAL NUMBERS

1.1 Set notation

A *set* is a collection of objects which are called its *elements*. If x is an element of the set S, we say that x *belongs* to S and write

$$x \in S.$$

If y does not belong to S, we write $y \notin S$.

The simplest way of specifying a set is by listing its elements. We use the notation

$$A = \{\tfrac{1}{2}, 1, \sqrt{2}, e, \pi\}$$

to denote the set whose elements are the real numbers $\frac{1}{2}, 1, \sqrt{2}, e$ and π. Similarly

$$B = \{\text{Romeo, Juliet}\}$$

denotes the set whose elements are Romeo and Juliet.

This notation is, of course, no use in specifying a set which has an infinite number of elements. Such sets may be specified by naming the property which distinguishes elements of the set from objects which are not in the set. For example, the notation

$$C = \{x : x > 0\}$$

(which should be read 'the set of all x such that $x > 0$') denotes the set of all positive real numbers. Similarly

$$D = \{y : y \text{ loves Romeo}\}$$

denotes the set of all people who love Romeo.

It is convenient to have a notation for the *empty* set $\emptyset$. This is the set which has *no* elements. For example, if x denotes a variable which ranges over the set of all real numbers, then

$$\{x : x^2 + 1 = 0\} = \emptyset.$$

This is because there are no real numbers x such that $x^2 = -1$.

If S and T are two sets, we say that S is a *subset* of T and write

$$S \subset T$$

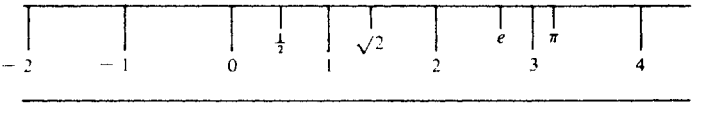
if every element of S is also an element of T.

As an example, consider the sets $P = \{1, 2, 3, 4\}$ and $Q = \{2, 4\}$. Then $Q \subset P$. Note that this is *not* the same thing as writing $Q \in P$, which means that Q is an element of P. The elements of P are simply 1, 2, 3 and 4. But Q is not one of these.

The sets A, B, C and D given above also provide some examples. We have $A \subset C$ and (presumably) $B \subset D$.

1.2 The set of real numbers

It will be adequate for this book to think of the real numbers as being points along a straight line which extends indefinitely in both directions. The line may then be regarded as an ideal ruler with which we may measure the lengths of line segments in Euclidean geometry.



The set of all real numbers will be denoted by $\mathbb{R}$. The table below distinguishes three important subsets of $\mathbb{R}$.

Subset	Notation	Elements
Natural numbers (or whole numbers)	$\mathbb{N}$	$1, 2, 3, 4, 5, \ldots$
Integers	$\mathbb{Z}$	$\ldots -2, -1, 0, 1, 2, 3, \ldots$
Rational numbers (or fractions)	$\mathbb{Q}$	$0, 1, 2, -1, \frac{1}{2}, \frac{3}{4}, \frac{5}{3}, -\frac{1}{2}, -\frac{3}{7}, \ldots$

Not all real numbers are rational. Some examples of irrational numbers are $\sqrt{2}$, e and π.

While we do not go back to first principles in this book, the treatment will be rigorous in so far as it goes. It is therefore important to be clear, at every stage, about what our assumptions are. We shall then know what has to be proved and what may be taken for granted. Our most vital assumptions are concerned with the properties of the real number system. The rest of this chapter and the following two chapters are consequently devoted to a description of the

properties of the real number system which we propose to assume and to some of their immediate consequences. A very much more systematic account of these assumptions is given in the author's book *Logic, Sets and Numbers* (see pp. 44–77).

1.3 Arithmetic

The first assumption is that the real numbers satisfy all the usual laws of addition, subtraction, multiplication and division.

The rules of arithmetic, of course, include the proviso that division by zero is not allowed. Thus, for example, the expression

$$\frac{2}{0}$$

makes no sense at all. In particular, it is *not* true that

$$\frac{2}{0} = \infty.$$

We shall have a great deal of use for the symbol ∞, but it must clearly be understood that ∞ does *not* represent a real number. Nor can it be treated as such except in very special circumstances.

1.4 Inequalities

The next assumptions concern inequalities between real numbers and their manipulation.

We assume that, given any two real numbers a and b, there are three mutually exclusive possibilities:

(i) $a > b$ (a is greater than b)

(ii) $a = b$ (a equals b)

(iii) $a < b$ (a is less than b).

Observe that $a < b$ means the same thing as $b > a$. We have, for example, the following inequalities.

$$1 > 0;\ 3 > 2;\ 2 < 3;\ -1 < 0;\ -3 < -2.$$

There is often some confusion about the statements

(iv) $a \geqslant b$ (a is greater than *or* equal to b)

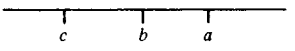
(v) $a \leqslant b$ (a is less than *or* equal to b).

To clear up this confusion, we note that the following are all true statements.

$$1 \geqslant 0;\ 3 \geqslant 2;\ 1 \geqslant 1;\ 2 \leqslant 3;\ -1 \leqslant 0;\ -3 \leqslant -3.$$

We assume four basic rules for the manipulation of inequalities. From these the other rules may be deduced.

(I) If $a > b$ and $b > c$, then $a > c$.



(II) If $a > b$ and c is any real number, then

$$a + c > b + c.$$

(III) If $a > b$ and $c > 0$, then $ac > bc$ (i.e. inequalities can be multiplied through by a *positive* factor).

(IV) If $a > b$ and $c < 0$, then $ac < bc$ (i.e. multiplication by a *negative* factor reverses the inequality).

1.5 *Example* If $a > 0$, prove that $a^{-1} > 0$.

Proof We argue by contradiction. Suppose that $a > 0$ but that $a^{-1} \leqslant 0$. It cannot be true that $a^{-1} = 0$ (since then $0 = 0.a = 1$). Hence

$$a^{-1} < 0.$$

By rule III we can multiply this inequality through by a (since $a > 0$). Hence

$$1 = a^{-1}.a < 0.a = 0.$$

But $1 < 0$ is a contradiction. Therefore the assumption $a^{-1} \leqslant 0$ was false. Hence $a^{-1} > 0$.

1.6 *Example* If x and y are positive, then $x < y$ if and only if $x^2 < y^2$.

Proof We have to show *two* things. First, that $x < y$ implies $x^2 < y^2$, and secondly, that $x^2 < y^2$ implies $x < y$.

(i) We begin by assuming that $x < y$ and try to deduce that $x^2 < y^2$. Multiply the inequality $x < y$ through by $x > 0$ (rule III). We obtain

$$x^2 < xy.$$

Similarly

$$xy < y^2.$$

But now $x^2 < y^2$ follows from rule I.

(ii) We now assume that $x^2 < y^2$ and try and deduce that $x < y$. Adding $-x^2$ to both sides of $x^2 < y^2$ (rule II), we obtain

$$y^2 - x^2 > 0$$

i.e. $$(y - x)(y + x) > 0. \qquad (1)$$

Since $x+y>0$, $(x+y)^{-1}>0$ (example 1.5). We can therefore multiply through inequality (1) by $(x+y)^{-1}$ to obtain

$$y-x>0$$

i.e. $$x<y.$$

(Alternatively, we could prove (ii) as follows. Assume that $x^2<y^2$ but that $x\geqslant y$. From $x\geqslant y$ it follows (as in (i)) that $x^2\geqslant y^2$, which is a contradiction.)

1.7 *Example* Suppose that, for *any* $\epsilon>0$, $a<b+\epsilon$. Then $a\leqslant b$.

Proof Assume that $a>b$. Then $a-b>0$. But, for *any* $\epsilon>0$, $a<b+\epsilon$. Hence $a<b+\epsilon$ in the particular case when $\epsilon=a-b$. Thus

$$a<b+(a-b)$$

and so $$a<a.$$

This is a contradiction. Hence our assumption $a>b$ must be false. Therefore $a\leqslant b$.

(Note: The symbol ϵ in this example is the Greek letter *epsilon*. It should be carefully distinguished from the 'belongs to' symbol $\in$ and also from the symbol ξ which is the Greek letter *xi*.)

1.8 *Exercise*

(1) If x is any real number, prove that $x^2\geqslant 0$. If $0<a<1$ and $b>1$, prove that

(i) $0<a^2<a<1$ (ii) $b^2>b>1$.

(2) If $b>0$ and $B>0$ and

$$\frac{a}{b}<\frac{A}{B},$$

prove that $aB<bA$. Deduce that

$$\frac{a}{b}<\frac{a+A}{b+B}<\frac{A}{B}.$$

(3) If $a>b$ and $c>d$, prove that $a+c>b+d$ (i.e. inequalities can be added). If, also, $b>0$ and $d>0$, prove that $ac>bd$ (i.e. inequalities between *positive* numbers can be multiplied).

(4) Show that each of the following inequalities may fail to hold even though $a>b$ and $c>d$.

(i) $a-c>b-d$

(ii) $\frac{a}{c} > \frac{b}{d}$

(iii) $ac > bd$.

What happens if we impose the extra condition that $b > 0$ and $d > 0$?

(5) Suppose that, for *any* $\epsilon > 0$, $a - \epsilon < b < a + \epsilon$. Prove that $a = b$.

(6) Suppose that $a < b$. Show that there exists a real number x satisfying $a < x < b$.

1.9 Roots

Let n be a natural number. The reader will be familiar with the notation $y = x^n$. For example, $x^2 = x.x$ and $x^3 = x.x.x$.

Our next assumption about the real number system is the following. Given any $y \geqslant 0$ there is exactly one value of $x \geqslant 0$ such that

$$y = x^n.$$

(Later on we shall see how this property may be deduced from the theory of continuous functions.)

If $y \geqslant 0$, the value of $x \geqslant 0$ which satisfies the equation $y = x^n$ is called the nth *root* of y and is denoted by

$$x = y^{1/n}.$$

When $n = 2$, we also use the notation $\sqrt{y} = y^{1/2}$. Note that, with this convention, it is always true that $\sqrt{y} \geqslant 0$. If $y > 0$, there are, of course, *two* numbers whose square is y. The positive one is $\sqrt{y}$ and the negative one is $-\sqrt{y}$. The notation $\pm\sqrt{y}$ means '$\sqrt{y}$ *or* $-\sqrt{y}$'.

If $r = m/n$ is a positive rational number and $y \geqslant 0$, we define

$$y^r = (y^m)^{1/n}.$$

If r is a negative rational, then $-r$ is a positive rational and hence y^{-r} is defined. If $y > 0$ we can therefore define y^r by

$$y^r = \frac{1}{y^{-r}}.$$

We also write $y^0 = 1$. With these conventions it follows that, if $y > 0$, then y^r is defined for all rational numbers r. (The definition of y^x when x is an irrational real number must wait until a later chapter.)

1.10 Quadratic equations

If $y > 0$, the equation $x^2 = y$ has two solutions. We denote the *positive* solution by $\sqrt{y}$. The *negative* solution is therefore $-\sqrt{y}$. We note again that

there is no ambiguity about these symbols and that $\pm\sqrt{y}$ simply means '$\sqrt{y}$ or $-\sqrt{y}$'.

The general quadratic equation has the form

$$ax^2 + bx + c = 0$$

where $a \neq 0$. Multiply through by $4a$. We obtain

$$4a^2x^2 + 4abx + 4ac = 0$$

$$(2ax + b)^2 - b^2 + 4ac = 0$$

$$(2ax + b)^2 = b^2 - 4ac.$$

It follows that the quadratic equation has no real solutions if $b^2 - 4ac < 0$, one real solution if $b^2 - 4ac = 0$ and two real solutions if $b^2 - 4ac > 0$. If $b^2 - 4ac \geqslant 0$,

$$2ax + b = \pm\sqrt{(b^2 - 4ac)}$$

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}.$$

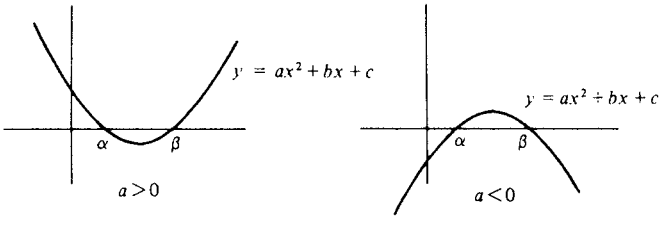
The roots of the equation $ax^2 + bx + c = 0$ are therefore

$$\alpha = \frac{-b - \sqrt{(b^2 - 4ac)}}{2a} \quad \text{and} \quad \beta = \frac{-b + \sqrt{(b^2 - 4ac)}}{2a}.$$

It is a simple matter to check that, for all values of x,

$$ax^2 + bx + c = a(x - \alpha)(x - \beta).$$

With the help of this formula, we can sketch the graph of the equation $y = ax^2 + bx + c$.



1.11 Example A nice application of the work on quadratic equations described above is the proof of the important *Cauchy–Schwarz inequality*. This asserts that, if $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are any real numbers, then

$$(a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 \leqslant (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2).$$

Proof For any x,

$$\begin{aligned} 0 &\leqslant (a_1x + b_1)^2 + (a_2x + b_2)^2 + \ldots + (a_nx + b_n)^2 \\ &= (a_1^2 + \ldots + a_n^2)x^2 + 2(a_1b_1 + \ldots + a_nb_n)x + (b_1^2 + \ldots + b_n^2) \\ &= Ax^2 + 2Bx + C. \end{aligned}$$

Since $y = Ax^2 + 2Bx + C \geqslant 0$ for *all* values of x, it follows that the equation $Ax^2 + 2Bx + C = 0$ cannot have two (distinct) roots. Hence

$$(2B)^2 - 4AC \leqslant 0$$

i.e. $$B^2 \leqslant AC$$

which is what we had to prove.

1.12 Exercise

(1) Suppose that n is an *even* natural number. Prove that the equation $x^n = y$ has no solutions if $y < 0$, one solution if $y = 0$ and two solutions if $y > 0$.

Suppose that n is an *odd* natural number. Prove that the equation $x^n = y$ always has one and only one solution.

Draw graphs of $y = x^2$ and $y = x^3$ to illustrate these results.

(2) Simplify the following expressions:

(i) $8^{2/3}$ (ii) $27^{-4/3}$ (iii) $32^{6/5}$.

(3) If $y > 0$, $z > 0$ and r and s are any rational numbers, prove the following:

(i) $y^{r+s} = y^r y^s$ (ii) $y^{rs} = (y^r)^s$ (iii) $(yz)^r = y^r z^r$.

(4) Suppose that $a > 0$ and that α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$ (in which $b^2 - 4ac > 0$). Prove that $y = ax^2 + bx + c$ is negative when $\alpha < x < \beta$ and positive when $x < \alpha$ or $x > \beta$. Show also (without the use of calculus) that $y = ax^2 + bx + c$ achieves a minimum value of $c - b^2/4a$ when $x = -b/2a$.

(5) Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Their arithmetic mean A_n and harmonic mean H_n are defined by

$$A_n = \frac{a_1 + a_2 + \ldots + a_n}{n} \qquad H_n^{-1} = \frac{1}{n}\left\{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right\}.$$

Deduce from the Cauchy–Schwarz inequality that $H_n \leqslant A_n$.

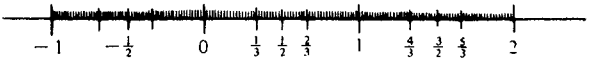
(6) Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any real numbers. Prove Minkowski's inequality, i.e.

$$\left\{\sum_{k=1}^{n} (a_k + b_k)^2\right\}^{1/2} \leqslant \left\{\sum_{k=1}^{n} a_k^2\right\}^{1/2} + \left\{\sum_{k=1}^{n} b_k^2\right\}^{1/2}.$$

For the case $n = 2$ (or $n = 3$) this inequality amounts to the assertion that the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides. Explain this.

1.13 Irrational numbers

In § 1.2 we mentioned the existence of irrational real numbers. That such numbers exist is by no means obvious. For example, one may imagine the process of marking all the rational numbers on a straight line. First one would mark the integers. Then one would move on to the multiples of $\frac{1}{2}$ and then to the multiples of $\frac{1}{3}$ and so on. Assuming that this program could ever be completed, one might very well be forgiven for supposing that there would be no room left for any more points on the line.



But our assumption about the existence of nth roots renders this view untenable. This assumption requires us to accept the existence of a positive real number x (namely $\sqrt{2}$) which satisfies $x^2 = 2$. If x were a rational number it would be expressible in the form

$$x = \frac{m}{n}$$

where m and n are natural numbers with no common divisor (other than 1). It follows that

$$m^2 = 2n^2$$

and so m^2 is even. This implies that m is even. (If m were odd, we should have $m = 2k + 1$. But then $m^2 = 4k^2 + 4k + 1$ which is odd.) We may therefore write $m = 2k$. Hence

$$4k^2 = 2n^2$$

$$n^2 = 2k^2.$$

Thus n is even. We have therefore shown that both m and n are divisible by 2. This is a contradiction and it follows that x cannot be rational, i.e. $\sqrt{2}$ must be an irrational real number.

Of course, $\sqrt{2}$ is not the only irrational number and the ability to extract nth roots allows us to construct many others. But it should not be supposed that all irrational numbers can be obtained in this way. It is not even true that every irrational number is a root of an equation of the form

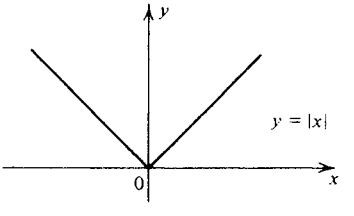
$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = 0$$

in which the coefficients $a_0, a_1, \ldots, a_n$ are rational numbers. Real numbers which are not the roots of such an equation are called transcendental, notable examples being e and π. This fascinating topic, however, lies outside the scope of this book.

1.14 Modulus

Suppose that x is a real number. Its *modulus* (or *absolute value*) $|x|$ is defined by

$$|x| = \begin{cases} x & (x \geqslant 0) \\ -x & (x < 0). \end{cases}$$



Thus $|3| = 3$, $|-6| = 6$ and $|0| = 0$. Obviously $|x| \geqslant 0$ for all values of x.

It is sometimes useful to note that $|x| = \sqrt{x^2}$.

1.15 *Theorem* For any real number x,

$$-|x| \leqslant x \leqslant |x|.$$

Proof Either $x \geqslant 0$ or $x < 0$. In the first case, $-|x| \leqslant 0 \leqslant x = |x|$. In the second case, $-|x| = x < 0 < |x|$.

1.16 *Theorem* For any real numbers a and b

$$|ab| = |a|.|b|$$

Proof The most elegant proof is the following.

$$|ab| = \sqrt{((ab)^2)} = \sqrt{(a^2b^2)} = \sqrt{(a^2)}.\sqrt{(b^2)} = |a|.|b|.$$

The next theorem is of great importance and is usually called the *triangle inequality*. This nomenclature may be justified by observing that the theorem is the special case of Minkowski's inequality (exercise 1.12(6)) with $n = 1$. We give a separate proof.

1.17 *Theorem (triangle inequality)* For any real numbers a and b,

$$|a + b| \leqslant |a| + |b|.$$

Proof We have

$$\begin{aligned} |a+b|^2 = (a+b)^2 &= a^2 + 2ab + b^2 \\ &= |a|^2 + 2ab + |b|^2 \\ &\leqslant |a|^2 + 2|ab| + |b|^2 \quad \text{(theorem 1.15)} \\ &= |a|^2 + 2|a| \cdot |b| + |b|^2 \quad \text{(theorem 1.16)} \\ &= (|a| + |b|)^2. \end{aligned}$$

It follows (from example 1.6) that

$$|a+b| \leqslant |a| + |b|.$$

1.18 *Theorem* For any real numbers c and d,

$$|c-d| \geqslant |c| - |d|.$$

Proof Take $a = d$ and $b = c - d$ in the triangle inequality. Then $a + b = c$ and so

$$|c| = |a+b| \leqslant |a| + |b| = |d| + |c-d|.$$

1.19 *Example*

(i) $2 = |-1+3| \leqslant |-1| + |3| = 1 + 3 = 4$.
(ii) $7 = |6-(-1)| \geqslant |6| - |-1| = 6 - 1 = 5$.

1.20 *Exercise*

(1) Prove that $|a| < b$ if and only if $-b < a < b$.
[Recall that you have *two* things to prove as in example 1.6.]

(2) Prove that

$$|c-d| \geqslant ||c| - |d||.$$

(3) The *distance* $d(x, y)$ between two real numbers x and y is defined by $d(x, y) = |x - y|$. Show that, for any x, y and z,
(i) $d(x, y) \geqslant 0$ (iii) $d(x, y) = 0$ if and only if $x = y$
(ii) $d(x, y) = d(y, x)$ (iv) $d(x, y) \leqslant d(x, z) + d(z, y)$.

† (4) If r and s are rational numbers, prove that

$$r + s\sqrt{2}$$

is irrational unless $s = 0$.

† (5) Suppose that the coefficients $a \neq 0, b$ and c of the quadratic equation

$$ax^2 + bx + c = 0$$

are all rational numbers and that $\alpha = r + s\sqrt{2}$ is a root of this equation, where r and s are also rational numbers. Prove that $\beta = r - s\sqrt{2}$ is also a root.

† (6) Prove that $3^{1/2}$ and $2^{1/3}$ are irrational numbers.

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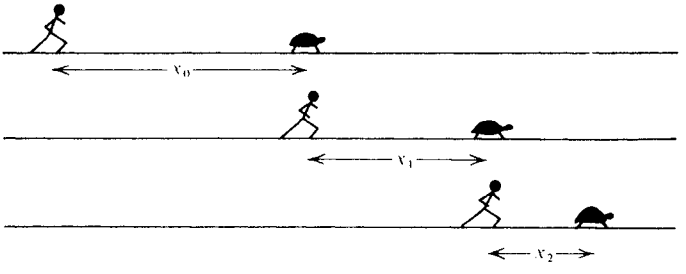
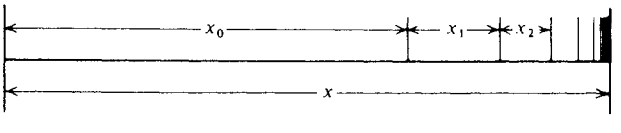
2 CONTINUUM PROPERTY

2.1 Achilles and the tortoise

The following is one of the famous paradoxes of Zeno. Achilles is to race a tortoise. Since Achilles runs faster than the tortoise, the tortoise is given a start of x_0 feet. When Achilles reaches the point where the tortoise started, the tortoise will have advanced a bit, say x_1 feet. Achilles soon reaches the tortoise's new position, but, by then, the tortoise will have advanced a little bit more, say x_2 feet. This argument may be continued indefinitely and so Achilles can never catch the tortoise.



The simplest way to resolve this paradox is to say that Achilles catches the tortoise after he has run a distance of x feet, where x is 'the smallest real number larger than all of the numbers $x_0, x_0+x_1, x_0+x_1+x_2 \ldots$.'. Zeno's argument then simply reduces to subdividing a line segment of length x into an infinite number of smaller line segments of respective lengths $x_0, x_1, x_2, \ldots$

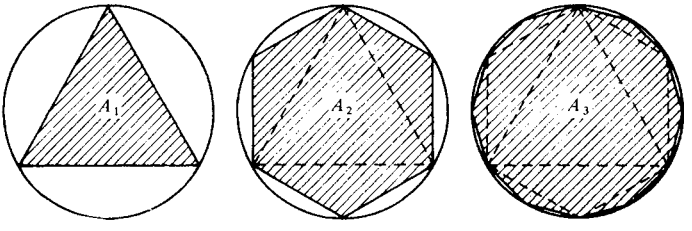


Formulated in this way, the paradox loses its sting.

This solution, of course, depends very strongly on the existence of the real number x, i.e. the smallest real number larger than all of the numbers x_0, $x_0 + x_1, x_0 + x_1 + x_2, \ldots$

A similar lesson is to be learned from Archimedes' method of exhaustion. This was invented by Archimedes as a means of evaluating the area of regions with curved boundaries.

In the case of a circle, the method involves inscribing a sequence of larger and larger regular polygons inside the circle and then calculating their areas $A_1, A_2, A_3, \ldots$



The area A of the circle is then identified with the 'smallest real number larger than each of the numbers $A_1, A_2, A_3, \ldots$'. By taking the radius of the circle to be 1 and calculating the value of A_n for a large enough value of n, Archimedes was able to obtain as close an approximation to π as he chose.

Note that this method again depends strongly on the existence of a smallest real number larger than all of the real numbers $A_1, A_2, A_3, \ldots$

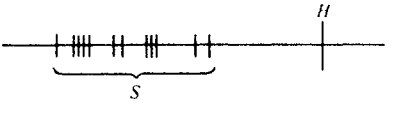
2.2 The Continuum Property

The discussion of the previous section indicates the necessity of making a further assumption about the real number system. It would, after all, be most unsatisfactory if Achilles never caught the tortoise or if a circle had no area. We shall call our new assumption the *Continuum Property* of the real numbers.

The Continuum Property is not usually mentioned explicitly in a school algebra course, but it is very important for what follows in these notes. The rest of this chapter is therefore devoted to its consideration. We begin with some terminology.

A set S of real numbers is *bounded above* if there exists a real number H which is greater than or equal to every element of the set, i.e. if, for some H,

$$x \leqslant H$$



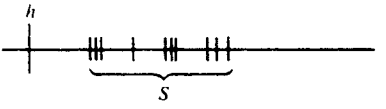
for any $x \in S$.

The number H (if such a number exists) is called an *upper bound* of the set S.

A set S of real numbers is *bounded below* if there exists a real number h

which is less than or equal to every element of the set, i.e. if, for some h,

$$x \geqslant h$$



for any $x \in S$.

The number h (if such a number exists) is called a *lower bound* of the set S.

A set which is both bounded above *and* bounded below is just said to be *bounded*.

2.3 ***Proposition*** A set S of real numbers is bounded if and only if there exists a real number K such that

$$|x| \leqslant K$$

for any $x \in S$. (This is easily proved with the help of exercise 1.20(1).)

2.4 *Examples*

(i) The set $\{1, 2, 3\}$ is bounded above. Some upper bounds are 100, 10, 4 and 3. The set is also bounded below. Some lower bounds are -27, 0 and 1.

(ii) The set $\{x : 1 \leqslant x < 2\}$ is bounded above. Some upper bounds are 100, 10, 4 and 2. The set is also bounded below. Some lower bounds are -27, 0 and 1.

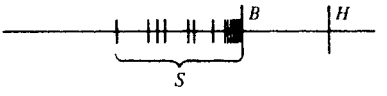
(iii) The set $\{x : x > 0\}$ is *unbounded above*. If $H > 0$ is proposed as an upper bound, one has only to point to $H + 1$ to obtain an element of the set larger than the supposed upper bound. However, the set $\{x : x > 0\}$ is bounded below. Some lower bounds are -27 and 0.

We now state the Continuum Property which will be fundamental for the remainder of these notes.

> *Continuum Property*
> Every non-empty set of real numbers which is bounded above has a smallest upper bound. Every non-empty set of real numbers which is bounded below has a largest lower bound.

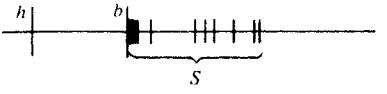
Thus, if S is a non-empty set which is bounded above, then S has an upper bound B such that, given any other upper bound H of S,

$B \leqslant H.$



Similarly, if S is a non-empty set which is bounded below, then S has a lower bound b such that, given any other lower bound h,

$b \geqslant h.$



2.5 Examples

(i) The smallest upper bound of the set $\{1, 2, 3\}$ is 3. The largest lower bound of the set $\{1, 2, 3\}$ is 1.

(ii) The smallest upper bound of the set $\{x : 1 \leqslant x < 2\}$ is 2. The largest lower bound is 1.

(iii) The set $\{x : x > 0\}$ has no upper bounds at all. The largest lower bound of the set $\{x : x > 0\}$ is 0.

2.6 Supremum and infimum

If a non-empty set S is bounded above, then, by the Continuum Property, it has a smallest upper bound B. This smallest upper bound B is sometimes called the *supremum* of the set S. We write $B = \sup S$ or

$$B = \sup_{x \in S} x.$$

Similarly, a set which is bounded below has a largest lower bound b. We call b the *infimum* of the set S and write $b = \inf S$ or

$$b = \inf_{x \in S} x.$$

Sometimes you may encounter the notation $\sup S = +\infty$. This simply means that S is unbounded above. Similarly, $\inf S = -\infty$ means that S is unbounded below.

2.7 Maximum and minimum

If a set S has a largest element M, we call M the *maximum* of the set S and write $M = \max S$. If S has a smallest element m, we call m the *minimum* of S and write $m = \min S$.

It is fairly obvious that, if a set S has a maximum M, then it is bounded above and its smallest upper bound is M. Thus, in this case, $\sup S = \max S$.

A common error is to suppose that the smallest upper bound of a set S is always the maximum of the set. However, some sets which are bounded above (and hence have a smallest upper bound) do *not* have a maximum. (See example 2.8(ii).)

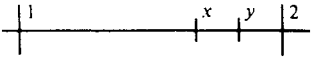
Similar remarks, of course, apply to largest lower bounds and minima of sets.

2.8 Examples

(i) The set $\{1, 2, 3\}$ has a maximum 3 and this is equal to its smallest upper bound. The set $\{1, 2, 3\}$ has minimum 1 and this is equal to its largest lower bound.

(ii) The set $\{x : 1 \leqslant x < 2\}$ has no maximum. The number 2 cannot be the largest element of the set because it does not belong to the set. On the other hand, any x in the set satisfies $1 \leqslant x < 2$. But then

$$y = \frac{x+2}{2}$$



is an element of the set which is larger than x. Hence x cannot be the largest element of the set. However, $\{x : 1 \leqslant x < 2\}$ has smallest upper bound 2.

The set $\{x : 1 \leqslant x < 2\}$ has minimum 1 (why?). This is equal to its largest lower bound.

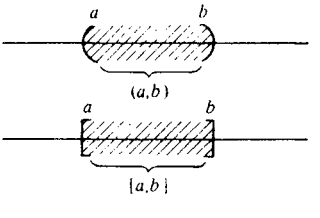
(iii) The set $\{x : x > 0\}$ has no maximum, nor does it have any upper bounds. The set $\{x : x > 0\}$ has no minimum (why not?). Its largest lower bound is 0.

2.9 Intervals

An *interval* I is a set of real numbers with the property that, if $x \in I$ and $y \in I$ and $x \leqslant z \leqslant y$, then $z \in I$, i.e. if two numbers belong to I, then so does every number between them.

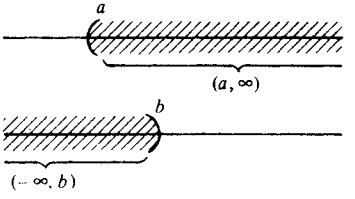
In describing intervals we use the following notation:

$$(a, b) = \{x : a < x < b\}$$
$$[a, b] = \{x : a \leqslant x \leqslant b\}$$
$$[a, b) = \{x : a \leqslant x < b\}$$
$$(a, b] = \{x : a < x \leqslant b\}.$$



These are the bounded intervals (classified by whether or not they have a maximum and whether or not they have a minimum). We also need to consider the unbounded intervals. For these we use the notation below.

$$(a, \infty) = \{x : x > a\}$$
$$[a, \infty) = \{x : x \geqslant a\}$$
$$(-\infty, b) = \{x : x < b\}$$
$$(-\infty, b] = \{x : x \leqslant b\}$$



(Do not be misled by this notation into supposing that ∞ or $-\infty$ are real numbers or that they can be treated as such.)

All intervals (with the exception of the empty set and the set of all real numbers) fall into one of the categories described above. We call the intervals (a, b), (a, ∞) and $(-\infty, b)$ *open* and the intervals $[a, b]$, $[a, \infty)$ and $(-\infty, b]$ *closed*. (The intervals $[a, b)$ and $(b, a]$ are sometimes called half-open.)

Of particular importance are the closed, bounded intervals $[a, b]$. We shall call such an interval *compact*.

2.10 Exercise

(1) Which of the following statements are true?
(i) $3 \in (1, 2)$ (ii) $2 \in (-\infty, 3]$ (iii) $1 \in [1, 2)$
(iv) $2 \in (0, 2)$ (v) $3 \in [1, 3]$.

(2) If ξ is a real number and $\delta > 0$, prove that

$$\{x : |\xi - x| < \delta\} = (\xi - \delta, \xi + \delta).$$

(3) Decide in each of the following cases whether or not the given set is bounded above. For those sets which are bounded above, write down three different upper bounds including the smallest upper bound. Decide which of the sets have a maximum and, where this exists, write down its value.
(i) $(0, 1)$ (ii) $(-\infty, 2]$ (iii) $\{-1, 0, 2, 5\}$
(iv) $(3, \infty)$ (v) $[0, 1]$.

(4) Repeat the above question but with the words 'above, upper and maximum' replaced by 'below, lower and minimum'.

(5) Give an example of a set which has smallest upper bound 4 but contains no element x satisfying $3 < x < 4$.

(6) Show that, given any element x of the set $(0, \infty)$, there is another element $y \in (0, \infty)$ with the property that $y < x$. Deduce that $(0, \infty)$ has no minimum.

2.11 Manipulations with sup and inf

We prove one result and list some others as exercises.

2.12 Theorem Suppose that S is a non-empty set of real numbers which is bounded above and $\xi > 0$. Then

$$\sup_{x \in S} \xi x = \xi \sup_{x \in S} x.$$

Proof Let $B = \sup S$. Then B is the smallest number such that, for any $x \in S$,

$$x \leqslant B.$$

Let $T = \{\xi x : x \in S\}$. Since $\xi > 0$,

$$\xi x \leqslant \xi B$$

for any $x \in S$. Hence T is bounded above by ξB. By the Continuum Property, T has a smallest upper bound (or supremum) C. We have to prove that $C = \xi B$. Since ξB is *an* upper bound for T and C is the *smallest* upper bound for T,

$$C \leqslant \xi B.$$

Now repeat the argument with the roles of S and T reversed. We know that C is the smallest number such that, for any $y \in T$,

$$y \leqslant C.$$

Since $\xi > 0$, it follows that

$$\xi^{-1} y \leqslant \xi^{-1} C$$

for any $y \in T$. But $S = \{\xi^{-1} y : y \in T\}$. Hence $\xi^{-1} C$ is *an* upper bound for S. But B is the *smallest* upper bound for S. Thus

$$B \leqslant \xi^{-1} C$$

$$\xi B \leqslant C.$$

We have shown that $C \leqslant \xi B$ and also that $\xi B \leqslant C$. Hence $\xi B = C$.

2.13 Exercise

(1) Suppose that S is bounded above and that $S_0 \subset S$. Prove that $\sup S_0 \leqslant \sup S$.

(2) Suppose that S is bounded above and that ξ is any real number. Prove that

$$\sup_{x \in S} (x + \xi) = \sup_{x \in S} x + \xi.$$

(3) Suppose that S is bounded above. Prove that

$$\inf_{x \in S} (-x) = -\sup_{x \in S} x.$$

Hence obtain analogies of theorem 2.12 and exercises 2.13(1 and 3) in which inf replaces sup.

(4) The distance $d(\xi, S)$ between a real number ξ and a non-empty set S of real numbers is defined by

$$d(\xi, S) = \inf_{x \in S} |\xi - x|$$

(see exercise 1.20(3)). Find the distance between the real number $\xi = 3$ and the sets

(i) $S = \{0, 1, 2\}$ (ii) $S = (0, 1)$

(iii) $S = [1, 2]$ (iv) $S = (2, 3)$.

(5) (i) If $\xi \in S$, prove that $d(\xi, S) = 0$. Give an example of a real number ξ and a non-empty set S for which $d(\xi, S) = 0$ but $\xi \notin S$.

(ii) If S is bounded above and $\xi = \sup S$, prove that $d(\xi, S) = 0$. Deduce that the same is true if S is bounded below and $\xi = \inf S$.

†(iii) If I is a closed interval, prove that $d(\xi, I) = 0$ implies that $\xi \in I$. If I is an open interval (other than $\mathbb{R}$ or $\emptyset$), show that a $\xi \notin I$ can always be found for which $d(\xi, I) = 0$.

*(6) Suppose that every element of an interval I belongs to one or the other of two non-empty subsets S and T. Show that one of the two sets S or T contains an element at zero distance from the other. (Do *not* assume that S and T are necessarily intervals. The set S, for example, could consist of all rational numbers in I and the set T of all irrational numbers in I.)

[Hint: Suppose that $s < t$, where $s \in S$ and $t \in T$. Let $T_0 = \{x : x \in T \text{ and } x > s\}$. Show that $T_0 \neq \emptyset$ and is bounded below. Write $b = \inf T_0$ and consider the two cases $b \in T$ and $b \notin T$.]

3 NATURAL NUMBERS

3.1 Introduction

In this chapter we explore some implications of the Continuum Property for sets of natural numbers. As always, arithmetical properties are taken for granted. But where a proof hinges on such a property, the property will be explicitly stated.

3.2 Archimedean property

The Archimedean property is the assertion of the following theorem.

3.3 *Theorem* The set $\mathbb{N}$ of natural numbers is unbounded above.

Proof For this proof we need to assume that, if n is a natural number, then so is $n + 1$.

Suppose that the theorem is false. Then $\mathbb{N}$ is bounded above. By the Continuum Property it therefore has a smallest upper bound B. Since B is the *smallest* upper bound for $\mathbb{N}$, $B - 1$ is *not* an upper bound for $\mathbb{N}$. Thus, for some $n \in \mathbb{N}$,

$$n > B - 1$$

$$\therefore \quad n + 1 > B.$$

But then $n + 1$ is an element of the set $\mathbb{N}$ which is greater than B. But B is an upper bound of the set $\mathbb{N}$. This is a contradiction.

Hence $\mathbb{N}$ is unbounded above.

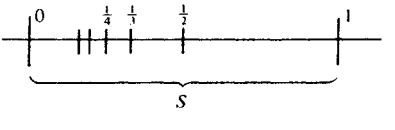
3.4 *Example* Prove that the set

$$S = \{n^{-1} : n \in \mathbb{N}\}$$

is bounded below with largest lower bound 0.

Proof Since the natural numbers are all positive,

$$n^{-1} > 0$$



for all $n \in \mathbb{N}$. Hence the set S is bounded below by 0. We have to prove that 0 is the *largest* lower bound.

Suppose that $h > 0$ is a lower bound for S. Then, for each $n \in \mathbb{N}$,

$$n^{-1} \geqslant h.$$

Hence, for each $n \in \mathbb{N}$,

$$n \leqslant h^{-1}.$$

But then h^{-1} is an upper bound for the set $\mathbb{N}$ which we know is unbounded above. It follows from this contradiction that no $h > 0$ can be a lower bound for S. Thus 0 is the largest lower bound for S.

Note that $0 \notin S$ and hence S has *no* minimum.

3.5 *Theorem* Every set of natural numbers which is not empty has a minimum.

Proof For this proof we need to assume that the distance between distinct natural numbers is at least 1.

Let S be a non-empty set of natural numbers. Given that all natural numbers are positive, 0 is a lower bound for S. By the Continuum Property, it follows that S has a largest lower bound b. Since b is the *largest* lower bound, $b + 1$ is *not* a lower bound. Therefore, for some $n \in S$,

$$n < b + 1.$$

If n is the minimum of S, there is nothing to prove. If not, then, for some $m \in S$, $m < n$. We obtain the inequality

$$b \leqslant m < n < b + 1$$

from which follows the contradiction $0 < n - m < 1$.

3.6 *Exercise*

(1) Prove that the set

$$S = \left\{ \frac{n-1}{n} : n \in \mathbb{N} \right\}$$

is bounded above with smallest upper bound 1. Does S have a maximum?

(2) Let $X > 1$. Prove that the set

$$S = \{X^n : n \in \mathbb{N}\}$$

is unbounded above. [Hint: if B were the smallest upper bound, then BX^{-1} could not be an upper bound.] Show that, if $0 < x < 1$, then the

set

$$T = \{x^n : n \in \mathbb{N}\}$$

is bounded below with largest lower bound 0.

(3) Prove that any non-empty set of integers which is bounded above has a maximum and that any non-empty set of integers which is bounded below has a minimum.

(4) Let $a < b$. Prove that there exists a *rational* number r satisfying $a < r < b$. [Hint: justify the existence of a natural number n satisfying $n > (b-a)^{-1}$ and consider the rational number $r = m/n$ where m is the smallest integer satisfying $m > an$.]

(5) Let S be the set of all *rational* numbers r which satisfy $0 < r < 1$. Show that S has no maximum and no minimum. Prove that S is bounded and has largest lower bound 0 and smallest upper bound 1. [Hint: for the last part, use the previous question.]

(6) Let $a < b$. Prove that there exists an *irrational* number ξ satisfying $a < \xi < b$. [Hint: exercise 1.20(4).]

3.7 Principle of induction

Suppose that a line of dominoes is arranged so that, if the nth one falls, it will knock over the $(n+1)$th. If the first domino is pushed over, most people would agree that *all* the dominoes would then fall down.

The principle of induction is an idealisation of this simple notion. We show below how it may be deduced from theorem 3.5.

3.8 Theorem (principle of induction)

Suppose that, for each $n \in \mathbb{N}$, $P(n)$ is a statement about the natural number n. Suppose also that

(i) $P(1)$ is true

(ii) if $P(n)$ is true, then $P(n+1)$ is true.

Then $P(n)$ is true for *every* $n \in \mathbb{N}$.

Proof For this proof we need to assume that, if n is a natural number other than 1, then $n-1$ is a natural number as well.

Let $S = \{n : P(n) \text{ is false}\}$. We want to prove that S is empty. Suppose that S is not empty. Then from theorem 3.5 it follows that S has a minimum m. Since $P(1)$ is true, $1 \notin S$. Hence $m \neq 1$. Therefore $m-1$ is a natural number. But m is the *minimum* of S. Thus $m-1 \notin S$ and so $P(m-1)$ is true. But then $P(m)$ is true (because of hypothesis (ii) of the theorem). Hence $m \notin S$ which is a contradiction.

3.9 *Example* For each $n \in \mathbb{N}$

$$1 + 2 + 3 + \ldots + n = \tfrac{1}{2}n(n+1).$$

Proof Let $S_n = 1 + 2 + 3 + \ldots + n = \Sigma_{k=1}^{n} k$. Let $P(n)$ be the statement $S_n = \frac{1}{2}n(n+1)$.

We have $S_1 = 1$, and $\frac{1}{2}.1(1+1) = 1$. Hence $P(1)$ is true. We now wish to show that 'If $P(n)$ is true, then $P(n+1)$ is true'. To do this, we assume that $P(n)$ is true and try and deduce that $P(n+1)$ is true. Thus we assume that $S_n = \frac{1}{2}n(n+1)$ and try and deduce that $S_{n+1} = \frac{1}{2}(n+1)(n+2)$. But

$$\begin{aligned} S_{n+1} &= 1 + 2 + 3 + \ldots + n + (n+1) \\ &= S_n + (n+1) \\ &= \tfrac{1}{2}n(n+1) + (n+1) \\ &= \tfrac{1}{2}(n+1)(n+2) \end{aligned}$$

as required. The result now follows by induction.

3.10 *Example* The real number

$$A_n = \frac{1}{n}(a_1 + a_2 + \ldots + a_n)$$

is the *arithmetic mean* of the numbers $a_1, a_2, \ldots, a_n$. If $a_1, a_2, \ldots, a_n$ are positive, their *geometric mean* G_n is defined by

$$G_n = (a_1 a_2 \ldots a_n)^{1/n}.$$

The 'inequality of the arithmetic and geometric means' asserts that

$$G_n \leqslant A_n.$$

Proof This inequality has an entertaining proof using 'backwards induction' (see exercise 3.11(5)).

Let $P(n)$ be the statement 'For any positive numbers $a_1, a_2, \ldots, a_n$, $G_n \leqslant A_n$'. We shall show that

(i) $P(2^n)$ is true for each $n \in \mathbb{N}$.

(ii) If $P(n)$ is true, then $P(n-1)$ is true.

The result then follows by backwards induction.

(i) We begin by proving $P(2)$. Now,

$$0 \leqslant (\sqrt{a_1} - \sqrt{a_2})^2 = a_1 - 2\sqrt{(a_1 a_2)} + a_2$$

$$(a_1 a_2)^{1/2} \leqslant \tfrac{1}{2}(a_1 + a_2).$$

This is simply $P(2)$. We now show that $P(2^n)$ implies $P(2^{n+1})$. Let $m = 2^n$. Then $2^{n+1} = 2m$. We have to assume $P(m)$ and try and deduce $P(2m)$. Since $P(m)$ is true,

$$(a_1 a_2 \ldots a_m)^{1/m} \leqslant \frac{1}{m}(a_1 + a_2 + \ldots + a_m)$$

and $$(a_{m+1} a_{m+2} \ldots a_{2m})^{1/m} \leqslant \frac{1}{m}(a_{m+1} + a_{m+2} + \ldots + a_{2m}).$$

But we know that $P(2)$ is true. Hence

$$\{(a_1 a_2 \ldots a_m)^{1/m}(a_{m+1} a_{m+2} \ldots a_{2m})^{1/m}\}^{1/2} \leqslant \frac{1}{2}\left\{\frac{a_1 + \ldots + a_m}{m} + \frac{a_{m+1} + \ldots + a_{2m}}{m}\right\}.$$

Thus

$$(a_1 a_2 \ldots a_{2m})^{1/2m} \leqslant \frac{1}{2m}(a_1 + a_2 + \ldots + a_{2m})$$

which is $P(2m)$.

We have shown that $P(2^1)$ is true and that, if $P(2^n)$ is true, then $P(2^{n+1})$ is true. Hence $P(2^n)$ is true for all $n \in \mathbb{N}$ by straightforward induction.

(ii) We now show that, if $P(n)$ is true, then $P(n-1)$ is true. Suppose that $P(n)$ is true. Then

$$\{a_1 a_2 \ldots a_{n-1} G_{n-1}\}^{1/n} \leqslant \frac{a_1 + a_2 + \ldots + a_{n-1} + G_{n-1}}{n}$$

$$\{G_{n-1}^{n-1} G_{n-1}\}^{1/n} \leqslant \frac{(n-1)A_{n-1} + G_{n-1}}{n}$$

$$nG_{n-1} \leqslant (n-1)A_{n-1} + G_{n-1}$$

$$G_{n-1} \leqslant A_{n-1}.$$

Thus $P(n-1)$ is true.

This completes the proof.

3.11 Exercise

(1) Prove by induction that

(i) $$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

(ii) $$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \tfrac{1}{4}n^2(n+1)^2.$$

(2) Prove by induction that, if $x \neq 1$,

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$